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Second-Order Tangent Structures

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Second-order differential processes have special significance for physics. Two reasonable generalizations of the procedure for constructing a tangent bundle over a smooth n-manifold M yield different second-order structures, each projecting onto the standard first-order structure TM. One approach, based on the work of Ehresmann generalizes the notion of a tangent vector as a derivation. The other, based on the work of Yano and Ishihara generalizes the notion of a tangent vector as the velocity of a curve. The former leads to J^2M, the 2-jet vector bundle consisting of second-order derivations, the latter leads to $T^{(2)}M$, the bundle of curves agreeing up to acceleration. Both project naturally onto TM because the 1-jet bundle of first-order derivations and the bundle of curves agreeing up to velocity are isomorphs of TM. Both generalizations admit extension to higher orders but the second-order case illustrates their differences and is important in applications. It is always true that J^2M is a vector bundle; but $T^{(2)}M$ is a vector bundle if and only if M has a linear connection and then $T^{(2)}M \equiv TM \oplus TM$ with fiber $\mathbf{R}^{2n}$, whereas J^2M always has fiber $\mathbf{R}^{(n^2+3n)/2}$. We compare these constructions and give some results about $T^{(2)}M$ and the principal bundle $L^{(2)}M$ to which it is associated. In a space-time there is a distinguished linear connection induced by the Lorentz metric, so both second-order tangent structures are available and the reduction of J^2M to $T^{(2)}M$ is a considerable simplification in the case $n=4$. We show also that both second-order bundles have applications to the study of space-time boundaries.

1. SECOND-ORDER TANGENT BUNDLES

We give a summary of the constructions in Ambrose et al. (1960) for J^2M and in Yano and Ishihara (1973) for $T^{(2)}M$ (cf. also Dodson and Radivoiovici, 1980).

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Take any $x \in M$. The *k-jet space*, $J_x^k M$, to M at x is the vector subspace of real-valued linear maps given by

$$J_x^k M = \left\{ v \in L(F_x; \mathbb{R}) \mid F_x^c \cup \left(F_x^0\right)^{k+1} \subseteq \ker v \right\}$$

where

$$F_x = \{ C^\infty f \colon N_x \to \mathbb{R} \mid N_x \text{ some open set about } x \}$$

$$F_x^c = \{ f \in F_x \mid f \text{ is constant on some open set about } x \}$$

$$F_x^0 = \{ f \in F_x \mid f(x) = 0 \}$$

$$\left(F_x^0\right)^{k+1} = \left\{ \text{finite sums of products of } (k+1) \text{ elements from } F_x^0 \right\}.$$

It follows immediately that $J_x^1 M$ is isomorphic to $T_x M$, the usual tangent space to M at x. Also, just as any chart (U, φ) with $x \in U$ determines a frame (∂_i) for $T_x M$ so also it determines a frame $(\partial_i, \partial_{ij} = \partial_{ji})$ for $J_x^2 M$. The vector spaces $J_x^2 M$ and $J_x^1 M$ collect into vector bundles $J^2 M$ and $J^1 M \equiv TM$ (cf. Palais, 1968).

Next we construct $T_x^{(2)} M$ from classes of curves in

$$C_x = \{ C^\infty f \colon (-\varepsilon, \varepsilon) \to M \mid f(0) = x, \text{ some real } \varepsilon > 0 \}$$

Such curves have first and second tangents, that is velocity and acceleration vectors, given by

$$\dot{f} \colon (-\varepsilon, \varepsilon) \to TM \colon t \rightsquigarrow D_t f(1)$$

$$\ddot{f} \colon (-\varepsilon, \varepsilon) \to TTM \colon t \rightsquigarrow D_t \dot{f}(1)$$

We introduce two equivalence relations $\sim_x$ and $\approx_x$ on C_x:

$$f \sim_x h \Leftrightarrow \dot{f}(0) = \dot{h}(0)$$

$$f \approx_x h \Leftrightarrow \dot{f}(0) = \dot{h}(0) \quad \text{and} \quad \ddot{f}(0) = \ddot{h}(0)$$

Then we can obtain vector spaces

$$C_x / \sim_x \equiv T_x M \equiv \mathbb{R}^n$$

$$C_x / \approx_x \equiv \mathbb{R}^{2n}$$

However, whereas we obtain from the former a vector bundle $T^{(1)}M$ naturally isomorphic to TM, it turns out that we can provide a vector bundle structure for $\{C_x/\approx_x | x\in M\}$ if and only if M has a linear connection (Dodson and Radivoiovici, 1980). In the latter case the bundle so formed is $T^{(2)}M \equiv TM\oplus TM$ and a chart (U,φ) determines a frame from $(\partial_i\oplus\partial_j)$ for $T_x^{(2)}M$ if it determines the frame (∂_i) for T_xM. In the work of Yano and Ishihara (1968, 1973) local properties were studied but the essential role of the connection was apparently not recognized (cf. Dodson and Radivoiovici, 1980; Radivoiovici, 1979).

We note that the manifolds that admit linear connections are precisely the paracompact ones (cf. Dodson, 1980, p. 147) and manifolds with connection form a full subcategory Man∇, of the category Man consisting of smooth manifolds and smooth maps. The tangent bundle functor, T: Man$\to$Vbun, induces functoriality of J^2 by putting $J^2\mu(\partial_i) = D\mu(\partial_i)$ in

$$J^2\colon \text{Man}\to\text{Vbun}\colon\quad \begin{matrix} M & & J^2M \\ \mu\downarrow & \longmapsto & \downarrow J^2\mu \\ M' & & J^2M' \end{matrix}$$

Similarly we obtain a functor

$$T^{(2)}\colon \text{Man}\nabla\to\text{Vbun}\colon\quad \begin{matrix} (M,\nabla) & & T^{(2)}M \\ \mu\downarrow & \longmapsto & \downarrow D^{(2)}\mu \\ (M',\nabla') & & T^{(2)}M' \end{matrix}$$

where

$$D^2\mu = F^{-1}\circ(D\mu\oplus D\mu)\circ F'$$

is well defined by the isomorphisms induced by connections (Dodson and Radivoiovici, 1980),

$$F\colon TM\oplus TM \equiv T^{(2)}M, \qquad F'\colon TM'\oplus TM' \equiv T^{(2)}M'$$

2. DISSECTION BY CONNECTION

One of the ways to view a linear connection ∇ is as a smooth splitting of TLM, the tangent bundle to the frame bundle, into horizontal and vertical components. This splitting induces corresponding splitting in J^2M and $T^{(2)}M$.

Take a chart (U, φ) on M and the induced frame field (∂_i); then ∇ appears locally in the form of a covariant derivation on vector fields given on generators by

$$\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$$

The splitting of J^2M, called a dissection in Ambrose et al. (1960), appears locally in the form

$$J^2U \equiv J^1U \oplus J^{\nabla}U$$
$$\left(v^i\partial_i + v^{ij}\partial_{ij}\right) \longmapsto \left(v^i + v^{jk}\Gamma_{jk}^i\right)\partial_i \oplus v^{ij}\left(\partial_{ij} - \Gamma_{ij}^k\partial_k\right)$$

and the members of $J^{\nabla}U$ are called pure second-order tangent vectors relative to ∇. In fact we can characterize $J^{\nabla}U$ as the kernel of a linear map

$$\chi: J^2U \to J^1M: \begin{cases} \partial_i \longmapsto \partial_i \\ \partial_{ij} \longmapsto \nabla_{\partial_i}\partial_j \end{cases}$$

Another characterization can be given by observing that at any $x \in M$ a complement of J_x^1M in J_x^2M is fixed by the spray of geodesics emanating from x. That was shown in Ambrose et al. (1960) and it was also shown that sprays and dissections are in one-to-one correspondence through the ability of either to determine a linear connection. Two connections, ∇ and ∇', generate the same geodesics, and hence the same sprays, if and only if their difference transformation is antisymmetric. That is, if and only if

$$(\forall u \in LM)(\forall X, Y \in T_u LM)$$

$$\delta(X)\theta(Y) = -\delta(Y)\theta(X)$$

where $\delta = \omega - \omega'$ is the difference between the two connection forms and θ is the canonical 1-form. This result follows from the structure equations (Ambrose, et al., 1960).

In a similar way we can represent the splitting of $T^{(2)}M$, which needs a connection to be a vector bundle at all, as a dissection given locally by (cf. Dodson and Radivoiovici, 1980).

$$T^{(2)}U \equiv TU \oplus TU$$

$$[f]_{\approx_x} \longmapsto \dot{f}(0) \oplus \nabla_{\dot{f}(0)}\dot{f}$$

Theorem 1. Each spray on M determines a unique linear connection (with any particular choice of torsion tensor) having geodesic spray the given spray by Ambrose et al. (1960), and hence it determines also a unique vector bundle structure on the acceleration bundle $T^{(2)}M$ by Dodson and Radivoiovici (1980). ■

We can see something more of the role of the connection in the following results.

Theorem 2. (i) There is a subcategory ManΔ of Man∇ consisting of the same objects as Man∇ but with morphisms

$$\mu: (M, \nabla) \to (M', \nabla') \qquad \text{in Man}\nabla$$

such that $\Delta\mu$ is in Vbun, where

$$\Delta\mu: T^{(2)}M \to T^{(2)}T': [f]_{\approx_x} \mapsto [\mu \circ f]_{\approx_{\mu(x)}}$$

(ii) $\Delta\mu = D^{(2)}\mu$ if and only if for all smooth curves $f: (-\varepsilon, \varepsilon) \to M$

$$D\mu\left(\nabla_{\dot{f}(0)}\dot{f}\right)(0) = \nabla'_{D\mu(\dot{f}(0))} D\mu(\dot{f})(0)$$

(iii) $T^{(2)}M \equiv T^{(2)}M'$ in Vbun if and only if there are isomorphisms A, B: $TM \equiv TM'$ in Vbun with

$$B\left(\nabla_{\dot{f}(0)}\dot{f}\right)(0) = \nabla'_{A\dot{f}(0)} A\dot{f}(0)$$

for all smooth curves $f: (-\varepsilon, \varepsilon) \to M$.

Proof. (i) This is a matter of checking the composition rule.

(ii) This is precisely the condition for $\Delta\mu$ to commute with the canonical projections from $T^{(2)}M$ and $T^{(2)}M'$ onto M and M'.

(iii) This follows from (ii) and the observation that the given equation implies commutativity with projections. ■

3. SECOND-ORDER PRINCIPAL BUNDLES

The tangent bundle TM is an associated bundle to the principal bundle LM with structure group $G^1(n) = Gl(n; \mathbb{R})$,

$$TM \equiv (LM \times \mathbb{R}^n)/G^1(n)$$

Similarly, we obtain (Dodson and Radivoiovici, 1980) a principal bundle $L^{(2)}M$ with

$$T^{(2)}M \equiv (L^{(2)}M \times \mathbb{R}^{2n})/G^1(2n) \qquad (\text{given } \nabla \text{ in } LM)$$

and LM is isomorphic to a subbundle of $L^{(2)}M$.

Working with k-jets, the principal bundle to which J^kM is associated is the Ehresmann (1953) bundle of k-frames P^kM with structure group $G^k(n)$ (cf. Kobayashi, 1972, p. 139). In particular, since $J^1M \equiv TM$ we find that also $P^1M \equiv LM$. We are only interested in the case $k=2$ and it is helpful to see P^2M constructed as follows.

Take any $x \in M$ and define on the set

$$E_xM = \{\text{local diffeos } f\colon (-\varepsilon, \varepsilon)^n \to M \mid f(0)=x, \text{ some real } \varepsilon>0\}$$

an equivalence relation (j_x^2) by

$$f(j_x^2)h \Leftrightarrow \begin{cases} \partial_i f = \partial_i h \text{ and } \partial_{ij} f = \partial_{ij} h, \text{ at } x \text{ for all } i, j \\ \text{with respect to the frame field } (\partial_i) \text{ of any chart.} \end{cases}$$

Then we take the set of equivalence classes

$$P_x^2M = E_xM/j_x^2 = \{[f]_{j_x^2} \mid f \in E_xM\}$$

and in fact it is usual to write $[f]_{j_x^2}$ as $j_x^2(f)$.

We construct $G^2(n)$ from similar classes, after replacing M by a neighborhood of the origin in $\mathbb{R}^n$:

$$G^2(n) = \{j_0^2(\sigma) \mid \sigma\colon (-\varepsilon, \varepsilon)^n \to (-\varepsilon', \varepsilon')^n, \text{ some real } \varepsilon, \varepsilon'>0\}$$

Then there is an obvious group structure given by

$$j_0^2(\sigma)\, j_0^2(\tau) = j_0^2(\sigma \circ \tau)$$

and the requisite free action on the right of $P^2M = \cup_{x \in M} P_x^2M$ is

$$P^2M \times G^2(n) \to P^2M\colon \left(j_x^2(f), j_0^2(\sigma)\right) \to j_x^2(f \circ \sigma)$$

Here we have followed Kobayashi (1961, 1972) but note that Hennig (1978) bases his representation of P^2M on classes of maps from M into $\mathbb{R}^n$, that is essentially a dual process to the one given above. Another view of P^2M is also useful; consider the following. Each $j_x^2(f) \in P^2M$ determines, with respect to chart (U, φ), a quadratic function f_φ of coordinates (x^i) by

$$f_\varphi: (-\varepsilon, \varepsilon)^n \to \varphi(U): (x^i) \rightsquigarrow \left(f^i + f_j^i x^j + \tfrac{1}{2} f_{jk}^i x^j x^k\right)$$

where $(f^i) = \varphi \circ f \circ \varphi^{-1}$, with ε small enough for imf $\subseteq U$,

$$f_j^i = \frac{\partial}{\partial x^j} f^i, \qquad f_{jk}^i = \frac{\partial}{\partial x^k} f_j^i$$

The coefficients in the quadratic f_φ serve as coordinates and in a similar way we have coordinates on $G^2(n)$, such that it appears as ordered arrays of real numbers. Namely,

$$G^2(n) = \left\{ \left(\sigma_j^i, \sigma_{jk}^i\right) \mid \det \sigma_j^i \neq 0, \sigma_{jk}^i = \sigma_{kj}^i \right\}$$

and its action on P^2M is given by

$$\left(f^i, f_j^i, f_{jk}^i\right) \times \left(\sigma_j^i, \sigma_{jk}^i\right) = \left(f^i, f_m^i \sigma_j^m, f_m^i \sigma_{jk}^m + f_{rs}^i \sigma_j^r \sigma_k^s\right)$$

Of course, if this change of 2-frame corresponds to a change of chart about some point $x \in U$ then σ_j^i and σ_{jk}^i are just

$$\frac{\partial x^i}{\partial y^j} \quad \text{and} \quad \frac{\partial^2 x^i}{\partial y^j \partial y^k}$$

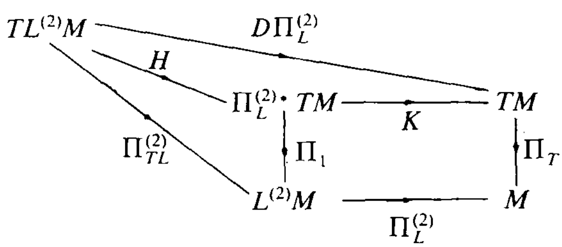
where (x^i) and (y^i) are the alternative coordinates about x.

We return to $L^{(2)}M$, which has a unique connection $\tilde{\nabla}$ from the connection ∇ in LM, as we have discussed elsewhere.

Theorem 3. $L^{(2)}M$ is parallelizable.

Proof. This parallels the proof that LM is parallelizable if it admits a connection and we give an outline only.

Since Vbun admits pullbacks we can construct $\Pi_L^{(2)*}M$, the pullback of TM over $L^{(2)}M$:



Then by the universal property of pullbacks, the canonical projections $D\Pi_L^{(2)}$ and $\Pi_{TL}^{(2)}$ determine a unique vector bundle morphism H, and also another, K, which gives linear isomorphisms on fibers. Specifically, we have a trivial bundle

$$\Pi_L^{(2)*}TM = \{(u, v) \in L^{(2)}M \times TM \mid \Pi_L^{(2)}(u) = \Pi_T(v)\}$$

Just as the connection ∇ in LM splits TLM so $\tilde{\nabla}$ in $L^{(2)}M$ splits each $T_u L^{(2)}M$ into $H_u \oplus G_u$, consisting of $\tilde{\nabla}$ horizontal and vertical members, so giving an exact sequence of vector spaces

$$0 \to G_u \to H_u \oplus G_u \to H_u \to 0$$

and hence an exact sequence of vector bundles

$$0 \to \ker D\Pi_L^{(2)} \to TL^{(2)}M \xrightarrow{H} \Pi_L^{(2)*}TM \to 0$$

Now, $\ker D\Pi_L^{(2)}$ is isomorphic to $L^{(2)}M \times G^1(2n)$ and therefore it is trivial and we can show that $TL^{(2)}M$ is the Whitney sum $\Pi_L^{(2)*}TM \oplus \ker D\Pi_L^{(2)}$. For, the $\tilde{\nabla}$-horizontal lifts (Dodson and Radivoiovici, 1980) yield a unique right inverse for H. Hence, $TL^{(2)}M$ is itself trivial and $L^{(2)}M$ is parallelizable. ■

Corollary. (i) $L^{(2)}M$ is orientable, metrizable, and admits a flat connection in $LL^{(2)}M$.

(ii) $L^{(2)}M$ admits a Riemannian structure with which the parallelization connection is compatible.

(iii) $L^{(2)}M \equiv L^{(2)}M' \Leftrightarrow T^{(2)}M' \equiv T^{(2)}M'$

Proof. These are standard deductions. ■

In the presence of a connection we may suppose that P^2M can itself be simplified. This is indeed the case and Kobayashi (1972) shows that a torsion-free linear connection ∇ on M corresponds precisely to a section

$$\nabla : M \to P^2M/G^1(n)$$

which is intuitively reasonable because such connections correspond to dissections of J^2M. Furthermore, if (θ_1, θ_2) is the canonical 1-form on P^2M and

$$\gamma : P^1M \hookrightarrow P^2M$$

is the injection arising from the section ∇ then

$$\theta = D\gamma(\theta_1) \text{ is the canonical 1-form on } P^1M = LM$$

$\omega = D\gamma(\theta_2)$ is the connection form corresponding to ∇.

Again there is a parallel situation for $L^{(2)}M$ which contains an isomorph of LM by means of

$$l: LM \to L^{(2)}M: (x, B) \to \left(x, \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \right)$$

and has a connection $\tilde{\nabla}$ induced by ∇. The corresponding canonical and connection forms $\tilde{\theta}$ and $\tilde{\omega}$ relate to those on LM by

$$\tilde{\theta} \circ (Dl) = (I, I) \circ \theta \quad \text{and} \quad \tilde{\omega} \circ Dl = \mathfrak{L} \circ \omega$$

where $\mathfrak{L}$ is the Lie algebra injection corresponding to l.

4. SPACE-TIME BOUNDARIES

A space-time (M, g) is a connected, noncompact, Hausdorff, inextensible smooth 4-manifold with a Lorentz structure. Hence, a space-time always has a unique torsion-free connection ∇ in LM, reducible to a connection in the pseudoorthonormal bundle OM with structure group $O(1,3)$. The central role of ∇ in relativity was extended by Schmidt (1971) to the characterization of singularities by incorporating them in the b-boundary ∂M of space-time. Details of this and subsequent modifications are given in Dodson (1978). We have shown (Dodson and Radivoiovici, 1981) that the connection $\tilde{\nabla}$ in $L^{(2)}M$ induced by ∇ in LM allows another view of singularities by means of the $\tilde{b}$-boundary $\tilde{\partial} M$, which contains ∂M. The intrinsic dependence of $\tilde{\partial} M$ on the acceleration of inextensible curves to which it supplies end points is attractive physically. For, the acceleration concerned is precisely the impediment to the curve developing as a geodesic; and for a physical particle it measures the external forces it experiences, that is its lack of freedom.

In the presence of a parallelization, that is a section of LM, an analytically simpler boundary for (M, g) can be constructed (Dodson and Sulley, 1980). Again, a similar construction can be applied to a section of $L^{(2)}M$. However, in either case some physical justification is required before invoking the extra structure that is needed. In the presence of ∇ in LM we have seen that $L^{(2)}M$ is naturally parallelizable through the existence of a section of $LL^{(2)}M$ because $TL^{(2)}M$ is trivial. For similar reasons LM is itself parallelizable without assuming extra structure for (M, g). Indeed the conformal structure (Kobayashi, 1972) induced by the given metric g was used by Schmidt (1974) to obtain a Riemannian structure on LLM from a parallelization and hence by projection he obtained a natural conformal boundary for a space-time. This procedure can be applied to obtain a Riemannian structure on $LL^{(2)}M$ and hence to obtain another conformal boundary.

For most realistic space-times it is quite likely that M is indeed parallelizable (cf. Dodson, 1980) but there is not a natural way to choose a section of LM to give a Riemannian structure. Similarly there may well exist sections of P^2M but none is distinguished physically. On the other hand, the Levi-Cività connection does determine a section of $P^2M/G^1(n)$, or its reduction corresponding to the replacement throughout of the general linear group by the Lorentz group. Hence we obtain a frame field for pure second-order tangent vectors which can be applied to curves, thus yielding another process for supplying acceleration-sensitive end points and hence another boundary for (M, g).

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